Math 1113: Solving equations involving trig or inverse trig functions

By Steve Fan

In this note, we discuss the general strategies for solving equations involving trig or inverse trig functions. We will focus on $\sin(x)$, $\cos(x)$, $\tan(x)$ and their corresponding inverse functions

$$\begin{aligned}
\arcsin(x) &= \sin^{-1}(x), \\
\arccos(x) &= \cos^{-1}(x), \\
\arctan(x) &= \tan^{-1}(x).
\end{aligned}$$

As for $\cot(x)$, $\sec(x)$ and $\csc(x)$, we can express these three functions in terms of $\sin(x)$, $\cos(x)$ and $\tan(x)$:

$$\begin{aligned}
\cot(x) &= \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}, \\
\sec(x) &= \frac{1}{\cos(x)}, \\
\csc(x) &= \frac{1}{\sin(x)}.
\end{aligned}$$

So equations involving these three functions can usually be transformed into equations concerning $\sin(x)$, $\cos(x)$, $\tan(x)$.

Keep in mind that although the strategies to be discussed below are applicable to a wide class of problems, you are advised to remain flexible in selecting methods. The unit circle and right triangles are also helpful. Some may be easier and more convenient to use than others in certain circumstances.

1 Trig equations

In this section, we discuss the strategy for solving equations of the form $f(Ax + B) = C$, where $f(x)$ is one of the three trig functions: $\sin(x)$, $\cos(x)$, $\tan(x)$, and A, B, C are constants with $A \neq 0$.

1.1 Solving $\sin(Ax + B) = C$

Below is a general procedure for finding all the solutions to the equation $\sin(Ax + B) = C$, where $-1 \leq C \leq 1$.

1. Find the angle θ in the first quadrant ($0 \leq \theta \leq \frac{\pi}{2}$) such that $\sin(\theta) = |C|$. Here $|C|$ is the absolute value of C. More often than not, the value of $|C|$ is one of the following:

$$0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1,$$

corresponding to the following angles

$$0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}.$$

2. Look at the sign of C.

   If $C \geq 0$, then the solutions are given by the equations $Ax + B = 2k\pi + \theta$ and $Ax + B = 2k\pi + (\pi - \theta)$, where $k = 0, \pm 1, \pm 2, \pm 3...$ is any integer.

   If $C < 0$, then the solutions are given by the equations $Ax + B = 2k\pi - \theta$ and $Ax + B = 2k\pi + (\pi + \theta)$, where, again, $k = 0, \pm 1, \pm 2, \pm 3...$ is any integer.

3. Solve for x.

   If $C \geq 0$, we solve $Ax + B = 2k\pi + \theta$ and $Ax + B = 2k\pi + (\pi - \theta)$ from Step 2. Moving B to the right-hand side and dividing dividing the equations by A, we get the solutions

   $$x = \frac{2k\pi + \theta - B}{A} \quad \text{and} \quad x = \frac{2k\pi + (\pi - \theta) - B}{A}.$$

   These are all the solutions to $\sin(Ax + B) = C$ when $C \geq 0$.

   If $C < 0$, we solve $Ax + B = 2k\pi - \theta$ and $Ax + B = 2k\pi + (\pi + \theta)$ from Step 2. As in the case $C \geq 0$, we get the solutions

   $$x = \frac{2k\pi - \theta - B}{A} \quad \text{and} \quad x = \frac{2k\pi + (\pi + \theta) - B}{A}.$$

   These are all the solutions to $\sin(Ax + B) = C$ when $C < 0$.

Step 2 of this algorithm can be explained as follows: In Step 1, we already find an angle $0 \leq \theta \leq \frac{\pi}{2}$ in the first quadrant such that $\sin(\theta) = |C|$. If $C \geq 0$, then $|C| = C$, so that $\sin(\theta) = C$. There is, however, another angle φ, lying in a different quadrant, with $\sin(\varphi) = C$. This angle lies in the second quadrant (recall that $\sin(x) \geq 0$ in the first and second quadrants and $\sin(x) \leq 0$ in the third and fourth quadrants), having $\varphi_R = \theta$, where φ_R is the reference angle of φ. So $\varphi = \pi - \varphi_R = \pi - \theta$. This explains the presence of θ and $\pi - \theta$ in Step 2. These two generate all the solutions to $\sin(x) = C$: we simply add $2k\pi$ to both θ and $\pi - \theta$ because $\sin(x)$ is periodic of period 2π.

On the other hand, if $C \leq 0$, then $|C| = -C$ and $\sin(-\theta) = -\sin(\theta) = -|C| = C$. In other words, $-\theta$ satisfies the equation $\sin(-\theta) = C$. Again, to find another one, we just do $\pi - (-\theta) = \pi + \theta$. This explains the presence of $-\theta$ and $\pi + \theta$ in Step 2. Then we add $2k\pi$

to both to get all solutions to $\sin(x)=C$.

Step 3 is merely about getting the solutions to $\sin(Ax+B)=C$ from those to $\sin(x)=C$ that we have found.

Example 1. Find all the solutions to $\sin(2x-1)=-\frac{\sqrt{3}}{2}$.

Solution. Step 1. Find the angle $0\le\theta\le\frac{\pi}{2}$ with $\sin(\theta)=\frac{\sqrt{3}}{2}$: $\theta=\frac{\pi}{3}$.

Step 2. Because $C=-\frac{\sqrt{3}}{2}<0$, we take $-\theta=-\frac{\pi}{3}$ and $\pi-(-\theta)=\pi+\theta=\frac{4\pi}{3}$ and add $2k\pi$ to both to get $2x-1=2k\pi-\frac{\pi}{3}$ and $2x-1=2k\pi+\frac{4\pi}{3}$.

Step 3. Solve for x: We solve $2x-1=2k\pi-\frac{\pi}{3}$ and $2x-1=2k\pi+\frac{4\pi}{3}$. The first equation gives

$$x=\frac{2k\pi-\frac{\pi}{3}+1}{2}=k\pi-\frac{\pi}{6}+\frac{1}{2},$$

while the second equation gives

$$x=\frac{2k\pi+\frac{4\pi}{3}+1}{2}=k\pi+\frac{2\pi}{3}+\frac{1}{2}.$$

Example 2. Find all the solutions x to $\sin(3x)=\frac{\sqrt{2}}{2}$ with x lying in the interval $[\frac{\pi}{6},\frac{5\pi}{6})$.

Solution. Step 1. Find the angle $0\le\theta\le\frac{\pi}{2}$ with $\sin(\theta)=\frac{\sqrt{2}}{2}$: $\theta=\frac{\pi}{4}$.

Step 2. Because $C=\frac{\sqrt{2}}{2}>0$, we take $\theta=\frac{\pi}{4}$ and $\pi-\theta=\frac{3\pi}{4}$ and add $2k\pi$ to both to get $3x=2k\pi+\frac{\pi}{4}$ and $3x=2k\pi+\frac{3\pi}{4}$.

Step 3. Solve for x: We solve $3x=2k\pi+\frac{\pi}{4}$ and $3x=2k\pi+\frac{3\pi}{4}$. The first equation gives

$$x=\frac{2k\pi+\frac{\pi}{4}}{3},$$

while the second equation gives

$$x=\frac{2k\pi+\frac{3\pi}{4}}{3}.$$

Step 4: There is an additional task for this particular problem: choose k such that the solutions are in $[\frac{\pi}{6},\frac{5\pi}{6})$. In general, this can always be done by setting

$$\frac{\pi}{6}\le\frac{2k\pi+\frac{\pi}{4}}{3}<\frac{5\pi}{6}$$

and

$$\frac{\pi}{6}\le\frac{2k\pi+\frac{3\pi}{4}}{3}<\frac{5\pi}{6}$$

and solving for k. Solving these two inequalities, we get

$$\frac{1}{8} \le k < \frac{9}{8}$$

and

$$-\frac{1}{8} \le k < \frac{7}{8}.$$

The only integer k between $\frac{1}{8}$ and $\frac{9}{8}$ is 1, and the only integer k between $-\frac{1}{8}$ and $\frac{7}{8}$ is 0. So we take $k = 1$ in

$$x = \frac{2k\pi + \frac{\pi}{4}}{3}$$

to get $x = \frac{3\pi}{4}$ and $k = 0$ in

$$x = \frac{2k\pi + \frac{3\pi}{4}}{3}$$

to get $x = \frac{\pi}{4}$. In other words, the solutions x to $\sin(3x) = \frac{\sqrt{2}}{2}$ with x lying in the interval $[\frac{\pi}{6}, \frac{5\pi}{6})$ are $x = \frac{3\pi}{4}$ and $x = \frac{\pi}{4}$.

1.2 Solving $\cos(Ax+B)=C$

Similarly, here's how we can find all the solutions to the equation $\cos(Ax+B)=C$, where $-1 \le C \le 1$.

1. Find the angle θ in the first quadrant ($0 \le \theta \le \frac{\pi}{2}$) such that $\cos(\theta) = |C|$. Again, the value of $|C|$ is often one of the following:

   $$0, \frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}, 1,$$

   corresponding to the following angles

   $$0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}.$$

2. Look at the sign of C.

   If $C \ge 0$, then the solutions are given by the equations $Ax + B = 2k\pi + \theta$ and $Ax + B = 2k\pi - \theta$.

   If $C < 0$, then the solutions are given by the equations $Ax + B = 2k\pi + (\pi + \theta)$ and $Ax + B = 2k\pi + (\pi - \theta)$.

3. Solve for x.

   If $C \geq 0$, we solve $Ax+B = 2k\pi+\theta$ and $Ax+B = 2k\pi-\theta$ from Step 2. Moving B to the right-hand side and dividing the equations by A, we get the solutions

   $$x = \frac{2k\pi+\theta-B}{A} \quad \text{and} \quad x = \frac{2k\pi-\theta-B}{A}.$$

   These are all the solutions to $\cos(Ax+B)=C$ when $C \geq 0$.

   If $C < 0$, we solve $Ax+B = 2k\pi+(\pi+\theta)$ and $Ax+B = 2k\pi+(\pi-\theta)$ from Step 2. As in the case $C \geq 0$, we get the solutions

   $$x = \frac{2k\pi+(\pi+\theta)-B}{A} \quad \text{and} \quad x = \frac{2k\pi+(\pi-\theta)-B}{A}.$$

   These are all the solutions to $\cos(Ax+B)=C$ when $C < 0$.

Example 3. Find all the solutions x to $2\cos\left(\frac{\pi}{2}x+\frac{\pi}{3}\right)+1=0$.

Solution. We move 1 to the right-hand side and divide the equation by 2:

$$\cos\left(\frac{\pi}{2}x+\frac{\pi}{3}\right) = -\frac{1}{2}$$

Step 1. Find the angle $0 \leq \theta \leq \frac{\pi}{2}$ with $\cos(\theta)=\frac{1}{2}$: $\theta = \frac{\pi}{3}$.

Step 2. Because $C = -\frac{1}{2} < 0$, we take $\pi+\theta = \frac{4\pi}{3}$ and $\pi-\theta = \frac{2\pi}{3}$ and add $2k\pi$ to both to get $\frac{\pi}{2}x+\frac{\pi}{3} = 2k\pi+\frac{4\pi}{3}$ and $\frac{\pi}{2}x+\frac{\pi}{3} = 2k\pi+\frac{2\pi}{3}$.

Step 3. Solve for x: We solve $\frac{\pi}{2}x+\frac{\pi}{3} = 2k\pi+\frac{4\pi}{3}$ and $\frac{\pi}{2}x+\frac{\pi}{3} = 2k\pi+\frac{2\pi}{3}$. The first equation gives

$$x = \frac{2k\pi+\frac{4\pi}{3}-\frac{\pi}{3}}{\frac{\pi}{2}} = 4k+2,$$

while the second equation gives

$$x = \frac{2k\pi+\frac{2\pi}{3}-\frac{\pi}{3}}{\frac{\pi}{2}} = 4k+\frac{2}{3}.$$

1.3 Solving $\tan(Ax+B)=C$

The procedure for solving $\tan(Ax+B)=C$ is much easier. Here's how it works:

1. Find the angle θ in the first quadrant ($0 < \theta < \frac{\pi}{2}$) such that $\tan(\theta) = |C|$. Oftentimes, the value of $|C|$ is one of the following:

$$0, \frac{\sqrt{3}}{3}, 1, \sqrt{3}$$

corresponding to the following angles

$$0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}.$$

2. Look at the sign of C.

If $C \geq 0$, then the solutions are given by the equation $Ax + B = k\pi + \theta$.

If $C < 0$, then the solutions are given by the equation $Ax + B = k\pi - \theta$.

3. Solve for x.

If $C \geq 0$, we solve $Ax + B = k\pi + \theta$ from Step 2. Moving B to the right-hand side and dividing the equation by A, we get the solutions

$$x = \frac{k\pi + \theta - B}{A}.$$

These are all the solutions to $\tan(Ax + B) = C$ when $C \geq 0$.

If $C < 0$, we solve $Ax + B = k\pi - \theta$ from Step 2. As in the case $C \geq 0$, we get the solutions

$$x = \frac{k\pi - \theta - B}{A}.$$

These are all the solutions to $\tan(Ax + B) = C$ when $C < 0$.

Example 4. Find all the solutions x to $\sin(\pi x) = \cos(\pi x)$.

Solution. We divide the equation $\sin(\pi x) = \cos(\pi x)$ by $\cos(\pi x)$ and use the definition

$$\tan(\pi x) = \frac{\sin(\pi x)}{\cos(\pi x)}$$

to get $\tan(\pi x) = 1$.

Step 1. Find the angle $0 \leq \theta \leq \frac{\pi}{2}$ with $\tan(\theta) = 1$: $\theta = \frac{\pi}{4}$.

Step 2. Because $C = 1 > 0$, we take $\theta = \frac{\pi}{4}$ and add $k\pi$ to it to obtain $\pi x = k\pi + \frac{\pi}{4}$.

Step 3. Solve for x: We solve $\pi x = k\pi + \frac{\pi}{4}$. Solving this equation for x, we get

$$x = k + \frac{1}{4}.$$

2 Inverse trig functions

In this section, we discuss the strategy for determining the values of the three inverse trig functions $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$, sometimes composed with the trig functions. Let's first recall the basic properties of these three functions.

1. $\arcsin(x)$ has domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. It is an odd function, meaning

$$\arcsin(-x) = -\arcsin(x).$$

   Looking at its graph, we see that $\arcsin(x)$ is increasing on its domain. In addition, we have the identities

$$\begin{aligned} \sin(\arcsin(x)) &= x, \qquad x \text{ in } [-1, 1], \\ \arcsin(\sin(x)) &= x, \qquad x \text{ in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{aligned}$$

2. $\arccos(x)$ has domain $[-1, 1]$ and range $[0, \pi]$. It satisfies

$$\arccos(-x) = \pi - \arccos(x).$$

   (This identity is equivalent to saying that $\arccos(x) - \frac{\pi}{2}$ is an odd function.) From its graph we can tell that $\arccos(x)$ is decreasing on its domain. Besides, we also have the identities

$$\begin{aligned} \cos(\arccos(x)) &= x, \qquad x \text{ in } [-1, 1], \\ \arccos(\cos(x)) &= x, \qquad x \text{ in } [0, \pi]. \end{aligned}$$

3. $\arctan(x)$ has domain $(-\infty, +\infty)$ and range $(-\frac{\pi}{2}, \frac{\pi}{2})$. It is an odd function, meaning

$$\arctan(-x) = -\arctan(x).$$

   The graph of $\arctan(x)$ shows that it is increasing on its domain. In addition, we have the identities

$$\begin{aligned} \tan(\arctan(x)) &= x, \qquad x \text{ in } (-\infty, +\infty), \\ \arctan(\tan(x)) &= x, \qquad x \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \end{aligned}$$

In the sequel, we shall use $f(x)$ and $g(x)$ to mean one of the three trig functions $\sin(x)$, $\cos(x)$, $\tan(x)$, and denote by $f^{-1}(x)$ and $g^{-1}(x)$ the corresponding inverse trig functions. For instance, if $f(x) = \sin x$ and $g(x) = \tan(x)$, then $f^{-1}(x) = \sin^{-1}(x) = \arcsin(x)$ and $g^{-1}(x) = \tan^{-1}(x) = \arctan(x)$.

2.1 Finding $f^{-1}(a)$

We start with the simplest problem: finding the value of $f^{-1}(a)$ with a given number a in the domain of $f^{-1}(x)$.

1. Set $\theta = f^{-1}(a)$.
2. The equation that we set up in Step 1 simply says that $f(\theta) = a$ with θ in the range of $f^{-1}(x)$.
3. Solve $f(\theta) = a$ for θ in the range of $f(x)$. (see the previous section on solving trig equations).

Example 5. Find $\arccos\left(-\frac{\sqrt{2}}{2}\right)$.

Solution. We follow the procedure described above.

Step 1. Set $\theta = \arccos\left(-\frac{\sqrt{2}}{2}\right)$.

Step 2. Get $\cos(\theta) = -\frac{\sqrt{2}}{2}$ with θ in the range $[0, \pi]$ of $\arccos(x)$.

Step 3. Solve $\cos(\theta) = -\frac{\sqrt{2}}{2}$ for θ in $[0, \pi]$, the interval covering the first and second quadrants: Because $\cos(x) \geq 0$ in the first quadrant and $\cos(x) \leq 0$ in the second quadrant, we know that $\frac{\pi}{2} \leq \theta \leq \pi$ (corresponding to the second quadrant). The only angle $\frac{\pi}{2} \leq \theta \leq \pi$ with $\cos(\theta) = -\frac{\sqrt{2}}{2}$ is $\theta = \frac{3\pi}{4}$. So $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

Example 6. Find $\arcsin\left(\frac{\tan(120°)}{2}\right)$.

Solution. We follow the same procedure as above.

Step 1. Set $\theta = \arcsin\left(\frac{\tan(120°)}{2}\right)$.

Step 2. Get $\sin(\theta) = \frac{\tan(120°)}{2}$ with θ in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of $\arcsin(x)$.

Step 3. Solve $\sin(\theta) = \frac{\tan(120°)}{2}$ with θ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the interval covering the first and fourth quadrants: Note that $\tan(120°) = -\sqrt{3}$, so that

$$\frac{\tan(120°)}{2} = -\frac{\sqrt{3}}{2}.$$

So $\sin(\theta) = \frac{\tan(120°)}{2} = -\frac{\sqrt{3}}{2}$. Because $\sin(x) \geq 0$ in the first quadrant and $\sin(x) \leq 0$ in the fourth quadrant, we know that $-\frac{\pi}{2} \leq \theta \leq 0$ (corresponding to the fourth quadrant). The only angle $-\frac{\pi}{2} \leq \theta \leq 0$ with $\sin(\theta) = -\frac{\sqrt{3}}{2}$ is $\theta = -\frac{\pi}{3}$. So $\arcsin\left(\frac{\tan(120°)}{2}\right) = -\frac{\pi}{3}$.

Example 7. Find $\sin(\arcsin(-0.5))$, $\arcsin\left(\sin\left(\frac{13\pi}{5}\right)\right)$, and $\arctan\left(\tan\left(-\frac{6\pi}{5}\right)\right)$.

Solution. We use the identities

$$\begin{aligned} \sin(\arcsin(x)) &= x, \qquad x \text{ in } [-1, 1], \\ \arcsin(\sin(x)) &= x, \qquad x \text{ in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \end{aligned}$$

and

$$\arctan(\tan(x)) = x, \qquad x \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

The first identity applied with $x = -0.5$ tells us that $\sin(\arcsin(-0.5)) = -0.5$.

To use the second identity for $\arcsin\left(\sin\left(\frac{13\pi}{5}\right)\right)$, we need to check whether $\frac{13\pi}{5}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. If it is, then the answer is simply $\frac{13\pi}{5}$. If not, we need to take $\frac{13\pi}{5}$ or $\pi - \frac{13\pi}{5} = -\frac{8\pi}{5}$ and add or subtract a suitable integer multiple of 2π to make either of the two in that interval, the result of which will be the answer. Observe that

$$-\frac{8\pi}{5} = \frac{2\pi}{5} - 2\pi$$

with $\frac{2\pi}{5}$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Now we can apply the second identity with $x = \frac{2\pi}{5}$ to get

$$\arcsin\left(\sin\left(\frac{13\pi}{5}\right)\right) = \arcsin\left(\sin\left(\frac{2\pi}{5}\right)\right) = \frac{2\pi}{5}.$$

This works because the periodicity of $\sin(x)$ (with period 2π) and the identity $\sin(\pi - x) = \sin(x)$ allow us to write

$$\sin\left(\frac{13\pi}{5}\right) = \sin\left(\pi - \frac{13\pi}{5}\right) = \sin\left(-\frac{8\pi}{5}\right) = \sin\left(2\pi - \frac{8\pi}{5}\right) = \sin\left(\frac{2\pi}{5}\right).$$

Similarly, we can use the third identity and the periodicity of $\tan(x)$ (with period π) to find

$$\arctan\left(\tan\left(-\frac{6\pi}{5}\right)\right) = \arctan\left(\tan\left(-\frac{\pi}{5} - \pi\right)\right) = \arctan\left(\tan\left(-\frac{\pi}{5}\right)\right) = -\frac{\pi}{5}.$$

2.2 Finding $f(g^{-1}(a))$

We conclude with a discussion on the more difficult problem: finding the value of $f(g^{-1}(a))$ with a given number a in the domain of $g^{-1}(x)$.

1. Set $\theta = g^{-1}(a)$.
2. The equation that we set up in Step 1 simply says that $g(\theta) = a$ with θ in the range of $g^{-1}(x)$.
3. Determine the value of $f(\theta)$ based on the given value $g(\theta) = a$.
4. Conclude that $f(g^{-1}(a)) = f(\theta)$.

Example 8. Find $\sin(\arccos(-\frac{4}{5}))$.

Solution. We follow the procedure as described above.

Step 1. Set $\theta = \arccos(-\frac{4}{5})$.

Step 2. Get $\cos(\theta) = -\frac{4}{5}$ with θ in the range $[0, \pi]$ of $\arccos(x)$.

Step 3. Determine the value of $\sin(\theta)$: We use the value $\cos(\theta) = -\frac{4}{5}$ with θ in $[0, \pi]$. If we look at the point $P(\cos(\theta), \sin(\theta))$ on the unit circle, then P lies in the second quadrant (corresponding to $\frac{\pi}{2} < \theta < \pi$), because θ is in $[0, \pi]$ with $\cos(\theta) < 0$. Using the reference angle θ_R of θ and the right triangle of side lengths $3, 4, 5$, with the side of length 3 facing θ_R (the sides $4, 5$ arise from $\cos(\theta) = -\frac{4}{5}$ and $3 = \sqrt{5^2 - 4^2}$ follows from the Pythagorean identity $a^2 + b^2 = c^2$ for a right triangle with sides a, b, c), we find that $\sin(\theta_R) = \frac{3}{5}$. Because $P(\cos(\theta), \sin(\theta))$ is in the second quadrant, we must have $\sin(\theta) > 0$. So $\sin(\theta) = \sin(\theta_R) = \frac{3}{5}$.

Alternatively, we can also use the Pythagorean identity

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

This gives

$$\sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)} = \pm\sqrt{1 - \left(-\frac{4}{5}\right)^2} = \pm\frac{3}{5}.$$

Again, because $P(\cos(\theta), \sin(\theta))$ is in the second quadrant, we know that $\sin(\theta) > 0$. So $\sin(\theta) = \frac{3}{5}$.

Step 4: We conclude that $\sin(\arccos(-\frac{4}{5})) = \sin(\theta) = \frac{3}{5}$.

Example 9. Find $\cos(\arctan(\frac{15}{8}))$.

Solution. We follow the same procedure as described above.

Step 1. Set $\theta = \arctan(\frac{15}{8})$.

Step 2. Get $\tan(\theta) = \frac{15}{8}$ with θ in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$ of $\arctan(x)$.

Step 3. Determine the value of $\cos(\theta)$: We use the value $\tan(\theta) = \frac{15}{8}$ with θ in $[0, \pi]$. If we look at the point $P(\cos(\theta), \sin(\theta))$ on the unit circle, then P lies in the first quadrant (corresponding to $0 < \theta < \frac{\pi}{2}$), because θ is in $(-\frac{\pi}{2}, \frac{\pi}{2})$ with $\tan(\theta) > 0$. Using the reference angle θ_R of θ and the right triangle of side lengths $8, 15, 17$, with the side of length 15 facing θ_R, we find that $\cos(\theta_R) = \frac{8}{17}$. Because $P(\cos(\theta), \sin(\theta))$ is in the first quadrant, we must have $\cos(\theta) > 0$. So $\cos(\theta) = \cos(\theta_R) = \frac{8}{17}$.

Alternatively, we can also use the Pythagorean identity

$$1 + \tan^2(\theta) = \sec^2(\theta).$$

This gives

$$\sec(\theta) = \pm\sqrt{1 + \tan^2(\theta)} = \pm\sqrt{1 + \left(\frac{15}{8}\right)^2} = \pm\frac{17}{8},$$

so that

$$\cos(\theta) = \frac{1}{\sec(\theta)} = \pm\frac{8}{17}$$

Again, because $P(\cos(\theta), \sin(\theta))$ is in the first quadrant, we know that $\cos(\theta) > 0$. So $\cos(\theta) = \frac{8}{17}$.

Step 4: We conclude that $\cos(\arctan(\frac{15}{8})) = \cos(\theta) = \frac{8}{17}$.

Example 10. Find $\tan(\arcsin(-\frac{12}{13}) - 3\pi)$.

Solution. First of all, we use the periodicity of $\tan(x)$ (with period π) to get rid of the -3π:

$$\tan\left(\arcsin\left(-\frac{12}{13}\right) - 3\pi\right) = \tan\left(\arcsin\left(-\frac{12}{13}\right)\right).$$

Now we follow the same procedure as above.

Step 1. Set $\theta = \arcsin(-\frac{12}{13})$.

Step 2. Get $\sin(\theta) = -\frac{12}{13}$ with θ in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of $\arcsin(x)$.

Step 3. Determine the value of $\tan(\theta)$: We use the value $\sin(\theta) = -\frac{12}{13}$ with θ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. If we look at the point $P(\cos(\theta), \sin(\theta))$ on the unit circle, then P lies in the fourth quadrant (corresponding to $-\frac{\pi}{2} < \theta < 0$), because θ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin(\theta) < 0$. Using the reference angle θ_R of θ and the right triangle of side lengths $5, 12, 13$, with the side of length 12 facing θ_R, we find that $\tan(\theta_R) = \frac{12}{5}$. Because $P(\cos(\theta), \sin(\theta))$ is in the second quadrant, we must have $\sin(\theta) < 0$ and $\cos(\theta) > 0$, so that

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} < 0.$$

So $\tan(\theta) = -\tan(\theta_R) = -\frac{12}{5}$.

Alternatively, we can also use the Pythagorean identity

$$1 + \cot^2(\theta) = \csc^2(\theta) = \left(\frac{1}{\sin(\theta)}\right)^2.$$

This gives

$$\cot(\theta) = \pm\sqrt{\left(\frac{1}{\sin(\theta)}\right)^2 - 1} = \pm\sqrt{\left(\frac{1}{-\frac{12}{13}}\right)^2 - 1} = \pm\sqrt{\left(-\frac{13}{12}\right)^2 - 1} = \pm\frac{5}{12},$$

so that

$$\tan(\theta) = \frac{1}{\cot(\theta)} = \pm\frac{12}{5}.$$

Again, because $P(\cos(\theta), \sin(\theta))$ is in the fourth quadrant, we know that $\tan(\theta) < 0$. So $\tan(\theta) = -\frac{12}{5}$.

Step 4: We conclude that

$$\tan\left(\arcsin\left(-\frac{12}{13}\right) - 3\pi\right) = \tan\left(\arcsin\left(-\frac{12}{13}\right)\right) = \tan(\theta) = -\frac{12}{5}.$$